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# ON THE FAMILY OF THE FIRST PASSAGE TIME OF SUMS OF VALUES OF FIRST ORDER AUTOREGRESSIVE PROCCES WITH RANDOM COEFFICIENT

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**Abstract.** In the paper the law of large numbers and central limit theorem for a family of passege time of Markov random walks described by the sum of values of first order autoregressive process with random coefficient are proved.

**Keywords:** First order autoregressive process with random coefficient, Law of large numbers, Central limit theorem, A family of first passege time.

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## 1. Introduction

Let $\xi_n, n \geq 1$ be a sequence of independent identically distributed random variables determined on the parabality space $(\Omega, \mathcal{F}, P)$.

Let us consider the first order autoregressive process $AR(1)$ in the following form

$$X_n - m = \beta(X_{n-1} - m) + \xi_n, \ n \geq 1, \quad (1)$$

where the initial value $X_0$ of the process is independent of the innovation $\{\xi_n\}$, and $m, \beta \in R = (-\infty, \infty)$ some fixed numbers.

The scheme $AR(1)$ in the form (1) has been considered in the works ([4], [5], [7], [8], [10], [13-16].

For the case of $AR(1)$ squence generated by the innovation with the normal distribution with parameters $(\theta, \sigma_2)$, the problems of testing statistical hypotheses with respect to the parameters $m, \beta, a$ and $\sigma^2$ have been studied in the paper [1-3,12].

To the specified work linear and nonlinear boundary problems for $AR(1)$ under different assumptions with respect to distribution of innovation have been studied for the case $m = 0$ .

At present great attention is paid to the study of limit theorems for family of the passige time of first order autoregressive process $(RCAR(1))$ with the random coefficient $\beta$.

The $RCAR(1)$ processes are first introduced and studied in the work [17] and [18].

Random coefficient autoregression processes have application in time series theory (see [18]).

In the present work we study linear boundary problems for the sum $L_n = \sum_{k=0}^{n} X_k$ in the case of $RCAR(1)$.

We will suppoce that the random variable $\beta$ is independent of random variables $X_0$ and in the independent of innnevation $\xi_n$.

## 2. Formulation and proof of main results

Assume

$$L_n^* = \frac{L_n - nm}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n} (X_k - m).$$

**Theorem 1**. Let $E\xi_1 = 0,\ D\xi_1 = \sigma^2\ E|X_0 - m| < \infty$ and for some $\varepsilon \in (0,1)$ $P(|\beta| < \varepsilon) = 1$.

Then

$$\lim_{n\to\infty} P((1-\beta)L_n^* \le x) = \Phi\left(\frac{x}{\sigma}\right), \quad x \in R,$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$.

**Proof.** From (1) we have

$$\sum_{k=1}^{n} (X_k - m) = \beta \sum_{k=1}^{n} (X_{k-1} - m) + \sum_{k=1}^{n} \xi_k$$

or

$$\sum_{k=0}^{n} (X_k - m) = \beta \sum_{k=1}^{n} (X_{k-1} - m) + X_0 - m + \sum_{k=1}^{n} \xi_k .$$

It is clear that

$$\sum_{k=0}^{n} (X_{k-1} - m) = \sum_{k=0}^{n-1} (X_k - m) = \sum_{k=0}^{n} (X_k - m) - (X_n - m).$$

Then we obtain

$$\sum_{k=0}^{n} (X_k - m) = \beta\left(\sum_{k=0}^{n} (X_k - m)\right) - \beta(X_n - m) + X_0 - m + \sum_{k=0}^{n} \xi_k$$

or

$$L_n^* = \beta L_n^* - \frac{\beta}{\sqrt{n}}(X_n - m) + \frac{X_0 - m}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \xi_k .$$

Hence we have

$$(1-\beta)L_n^* = \frac{\beta}{\sqrt{n}}(X_n - m) + \frac{X_0 - m}{\sqrt{n}} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\xi_k \; . \tag{2}$$

The condition $E|X_0|^2 < \infty$ and Markov inequality yield that

$$\frac{X_0 - m}{n} \to 0 \quad as \quad n \to \infty . \tag{3}$$

Show that provided $P(|\beta| < \varepsilon) = 1$ there exists a unique stationary solution of the equation (1) refresented in the form

$$X_n = m + \sum_{i=0}^{\infty} \beta^i \xi_{n-i} \; . \tag{4}$$

The indicated series converges in mean square sence.

It is easy to check that (4) is the solution of the equation (1). From (1) by means successive iterations we can obtain the following representation for

$$X_n - m = \beta(X_{n-1} - m) + \xi_n = \beta(\beta(X_{n-2} - m) + \xi_{n-1}) + \xi_n =$$

$$= \beta^n (X_0 - m) + \sum_{i=0}^{n-1} \beta^i (\xi_{n-i}) . \tag{5}$$

From this equality we have

$$E\left( X_n - m - \sum_{i=0}^{n-1} \beta^i (\xi_{n-i}) \right)^2 = E\left(\beta^n (X_0 - m)\right)^2 =$$

$$= E\beta^{2n} E(X_0 - m)^2 \to 0 \quad n \to \infty$$

since

$$E|X_0 - m|^2 < \infty$$

and

$$E\beta^{2n} \le \varepsilon^{2n} \to 0 \;\; as \;\; n \to \infty .$$

Thus, (4) is a unique solution of the equation (1) in the class of stationary sequences with a finite second moment.

We have show that in the right side of the equality (2) the first term converges zero in probability

$$\frac{\beta}{\sqrt{n}}(\xi_n - m) \xrightarrow{P} 0, \quad as \quad n \to \infty . \tag{6}$$

Indeed, by virtue of the Markov inequality we have

$$P\left(|X_n - n| > \varepsilon\sqrt{n}\right) \le \frac{E|X_n - n|^2}{n} . \tag{7}$$

From (5), by virtue of independence of random variables $\beta$ and $\xi_n$ and $E\xi_n = 0$ .

$$E|X_n - m|^2 = E\beta^{2n}|X_0 - m|^2 + \sum_{i=0}^{n-1} E\beta^{2i}\xi_{n-i}^2 =$$

$$= E\beta^{2n} E\left|X_0 - m\right|^2 + \sigma^2 E \sum_{i=1}^{n-1} \beta^{2i} \to \sigma^2 E \frac{1}{1-\beta^2} < \frac{\sigma^2}{1-\varepsilon^2}.$$

Therefore, (6) follows from (7).

For the last term in the right side of (2), according to the central limit theorem we have

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \xi_k \le k\right) = \Phi\left(\frac{x}{\sigma}\right). \tag{8}$$

Then from (3), (6) and (8) we complete the proof of theorem 1.

By means of theorem 1, we can study asymptotic behavior of a family of the moments of the first intersection

$$\tau_a = \inf\{n \ge 0 \,:\, L_n \ge a\} \tag{9}$$

of the level $a \ge 0$ by the sum $L_n = \sum_{k=0}^{n} X_k,\ n \ge 1$.

Similar problems have been studied in [14-16] the case when the coefficient $\beta$ in the model (1) is not a random value.

We have

**Theorem 2.** Let conditions of Theorem 1 be fulfilled, and suppose that $0 < m < \infty$ and $as \quad n \to \infty$

$$\frac{1}{n} \sum_{k=0}^{n} X_k \xrightarrow{a.s.} m \quad . \tag{10}$$

Then

1) $P(\tau_a < \infty) = 1$ for all $a \ge 0$,

2) $\tau_a \xrightarrow{a.s.} \infty, \quad as \quad a \to \infty$,

3) $\dfrac{\tau_a}{a} \xrightarrow{a.s.} \dfrac{1}{m}, \quad as \quad n \to \infty$

**Proof.** By the definition of the variable $\tau_a$ it is easy to be convinced that the following equality is fulfilled:

$$\{\omega : \tau_a < \infty\} = \left\{\omega : \sup_{n\ge 0} L_n \ge a\right\}. \tag{11}$$

Condition (10) fields

$$P\left(\omega : \sup_{n\ge 0} L_n = \infty\right) = 1.$$

Then statement 1) follows from (11).

It is clear that for $n \ge 0$ we have

$$P(\tau_a > n) = P\left(\sup_{0\le k\le n} L_k < a\right).$$

Therefore, for each $n \ge 0$

$$\lim_{a\to\infty} P(\tau_a > n) = 1.$$

This means that $\tau_a \xrightarrow{P} \infty, \quad as \quad a \to \infty$.

It is easy to understand that the variables $\tau$ increase as functions of $a$. Therefore statement 2) is valid.

Prove statement 3). By the definition of the variable $\tau_a$ we can write

$$\frac{L_{\tau_a - 1}}{\tau_a} < \frac{a}{\tau_a} \le \frac{L_{\tau_a}}{\tau_a} . \tag{12}$$

Prove that

$$\frac{L_{\tau_a}}{\tau_a} \xrightarrow{a.s.} m, \quad \text{as} \quad a \to \infty . \tag{13}$$

For this we need the following fact formulated as a lemma.

**Lemma 1.** Let the squence of random variables $\eta_n$ almost surely converge to the random variable $\eta$ and the family of integral-valued non-negative random variables $t_a,\ a \ge 0$ almost surely converge to infinity $t_a \xrightarrow{a.s.} \infty$ as $a \to \infty$. Then $\eta_{t_a} \xrightarrow{a.s.} \eta$ as $a \to \infty$.

**Proof.** Assume

$$A = \left\{\omega : \eta \xrightarrow[n \to \infty]{} \infty\right\}$$

$$B = \left\{\omega : t_a \xrightarrow[a \to \infty]{} \infty\right\}$$

$$C = \left\{\omega : \eta_{t_a} \xrightarrow[a \to \infty]{} \eta\right\} .$$

By the conditions of the lemma $P(A) = P(B) = 1$.

It is clear that the event $C$ will happen if events $A$ an $B$ happen simultemeously, i.e. $AB \subset C$.

Then, from the obvious inequality

$$P(AB) \ge P(A) + P(B) - 1 = 1 ,$$

we obtain $P(C) = 1$.

Lemma 1 is proved.

Convergence (13) follows by means of lemma 1 from condition (10) and statement 1) of the theorem proved.

**Theorem 3.** Let the conditions of Theorem 2 be fulfilled, and

$$\frac{X_n}{\sqrt{n}} \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty . \tag{14}$$

Then

$$P\left((1 - \beta)\tau_a^* \le x\right) = \Phi\left(\frac{x}{\sigma}\right),$$

where $\tau_a^* = \dfrac{\tau_a - \dfrac{a}{m}}{m^{-\frac{3}{2}}\sqrt{a}}$.

**Proof.** Assume $R_a = L_{\tau_a} - a$.

Prove that

$$(1-\beta)L_{\tau_a}^* = (1-\beta)\frac{L_{\tau_a} - m\tau_a}{\sqrt{\tau_a}} =$$

$$= (1-\beta)\frac{a - m\tau_a}{\sqrt{\tau_a}} + (1-\beta)\frac{R_a}{\sqrt{\tau_a}} =$$

$$= (\beta-1)\frac{\tau_a - \dfrac{a}{m}}{\sqrt{\tau_a}} + (1-\beta)\frac{R_a}{\sqrt{\tau_a}}. \tag{15}$$

Prove that

$$\lim_{a\to\infty} P\left((1-\beta)L_{\tau_a}^* \le x\right) = \Phi\left(\frac{x}{\sigma}\right). \tag{16}$$

For that, according to theorem 1, by virtue of statement 3) of theorem 1 and Anscombe theorem of sufies to show that the sequence $L_n^* = \dfrac{L_n - nm}{\sqrt{n}},\ n \ge 1$ uniformly continuous in parability. To this end, we will need the following known facts [6], [19] formulated in the form of a lemma.

**Lemma.** Let the sequence of random variables $Y_n,\ n \ge 1$ converge in distribution to the random variable $Y\left(Y_n \xrightarrow{d} Y\right)$ and be uniformly continuous in probability, i.e. the following relation be fulfilled for:

$$\lim_{\rho\to 0}\sup_{n\ge 1} P\left(\max_{1\le k\le np}\left|Y_{n+k} - Y_n\right| \ge \varepsilon\right) = 0 \tag{17}$$

for any $\varepsilon > 0$.

Furthermore, let $N(t),\ t > 0$ best family of integral valued non-negative random variables that $\dfrac{N(t)}{t} \xrightarrow{P} c$, $t \to \infty$ $c > 0$ is some constant.

Then $Y_{N(t)} \xrightarrow{d} Y,\ as\ t \to \infty$. This lemma is one of the variants of the Auscombe theorem (see [6], [19]).

**Lemma 2.** Note that from the Cauchy criterion on almost convergence in follows that $Y_{N(t)} \xrightarrow{a.s.} Y,\ as\ t \to \infty$ convergence yields (17) (see. [19], p. 41).

The property of uniform continuity of the sequence $L_n^*,\ n \geq 2$ follows from the equality (2) and condition (14), since it is known well that the squence $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \xi_k$ , $n \geq 1$ is uniformily continuous in probability [19].

Then, by the statement 3) of Theorem 2 and lemma of Theorem 1 we have (16).

We now prove that in the equality (15)

$$\frac{R_a}{\sqrt{\tau_a}} \xrightarrow{P} 0, \quad as \quad n \to \infty . \tag{18}$$

By the definition of the variable $\tau_a$ we have

$$0 \leq R_a = L_{\tau_a} - a < L_{\tau_a} - L_{\tau_a - 1} = X_{\tau_a}$$

since $0 < L_{\tau_a - 1}$.

Hence it is that in order to prove (18), it suffies to show that

$$\frac{X_{\tau_a}}{\sqrt{\tau_a}} \xrightarrow{P} 0, \quad a \to \infty . \tag{19}$$

To this end, we show that the squence $\frac{X_n}{\sqrt{n}}$ , $n \geq 1$ is uniformly continuous in probability.

Indeed, we have

$$\frac{X_n}{\sqrt{n}} = \frac{L_n - L_{n-1}}{\sqrt{n}} = \frac{L_n - mn}{\sqrt{n}} - \frac{L_n - (n-1)m}{\sqrt{n}} - \frac{m}{\sqrt{n}} =$$

$$= L_n^* - \sqrt{\frac{n-1}{n}} L_{n-1}^* - \frac{m}{\sqrt{n}} .$$

Hence it is seen that the squence $\frac{X_n}{\sqrt{n}}$ , $n \geq 1$ is uniformly continuous in probability due to lemma 2.

Then (10) follows by virtue of (6), lemma 1.

Thus, from (15), (16) and (18) we have

$$\lim_{a \to \infty} P\left( (\beta - 1) \frac{\tau_a - a/m}{\frac{1}{m}\sqrt{\tau_a}} \leq x \right) = \Phi\left(\frac{x}{\sigma}\right). \tag{20}$$

Hence, taking into account statement 3) of theorem 2 and equality $\Phi(x) = 1 - \Phi(-x)$ we complete the proof of theorem 3.

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